Abstract: In this paper, a new image denoising method based on wavelet analysis and support vector machine regression (SVR) is presented. The feasibility of image denoising via support vector regression is discussed and an illustrative example is given. The wavelet kernel is proposed to construct wavelet support vector machine (WSVM). The result of experiment shows that the denoising method based on WSVM can reduce noise better than Gaussian KBF SVM and other traditional methods.

Key words: Image denoising, Support vector regression, Wavelet analysis, Function approximation

1. Introduction

The support vector machine (SVM) is a new machine learning theory based on statistic learning theory proposed by Vladimir N. Vapnik[1]. It has been widely applied to pattern recognition, function approximation and system identification because SVM is able to deal with classification (SVC) and regression (SVR) problems. In this paper, support vector regression (SVR) is proposed to approximate an image as a 2 dimensional continuous function. Wavelet analysis is discussed and we construct the wavelet support vector machine (WSVM) to approximate image instead of traditional SVM, The ability of WSVM to reduce the noise is compared with traditional method. In section 2, the support vector regression is briefly reviewed. Section 3 discusses the feasibility of denoising based on SVR. Section 4 introduces the wavelet theory and proposed the wavelet kernel that has better ability to approximate complex nonlinear function than traditional kernels, and then the WSVM is constructed with wavelet kernel. In section 5, some illustrative results for image denoising are given. The comparison with Gaussian KBF SVM and other method are also discussed. Section 6 comes to the conclusion.

2. Review of support vector regression(SVR)

A brief review for SVR is given below. There is more detailed description of SVR In [2]. Let us consider the regression in the set of functions

\[ f(w) = w^T \varphi(x) + b \] (1)
Given training data \( \{x_i, y_i, i = 1, ..., N\} \), where \( w \in R^{mh} \) is the weight vector, \( N \) denote the number of training data, \( x \in R^m \) are the input data, \( y \in R \) are the output data, and \( b \in R \), \( \varphi(x) \) mapping the input data into a higher dimensional feature space, where \( \varphi(\bullet) : R \rightarrow R^{mh} \). In the support vector method one aims at minimizing the empirical risk

\[
R_{emp}(w, b) = \frac{1}{N} \sum_{i=1}^{N} |y_i - w^T \varphi(x_i) - b|
\]  

subject to \( w^T w < c \). The loss function employs Vapnik’s \( \varepsilon \)-insensitive model [1], and the function estimation problem is formulated then as

\[
\min \frac{1}{2} w^T w + c \left\{ \sum_{i=1}^{N} \xi^* + \sum_{i=1}^{N} \xi \right\}
\]

subject to the constraints

\[
\begin{align*}
  y_i - w^T \varphi(x_i) - b &\leq \varepsilon + \xi^*, i = 1, ..., N \\
  -y_i + w^T \varphi(x_i) + b &\leq \varepsilon + \xi, i = 1, ..., N \\
  \xi^* &\geq 0, i = 1, ..., N \\
  \xi &\geq 0, i = 1, ..., N
\end{align*}
\]

where \( \xi \) and \( \xi^* \) are the slack variables, \( c \) is a positive real constant and \( \varepsilon \) is the insensitivity region. The solution is given by

\[
w = \sum_{i=1}^{N} (\alpha^* - \alpha) \varphi(x_i)
\]

where \( \alpha^* \) and \( \alpha \) are the Lagrange multipliers. The dimension of feature space does not have to be specified because of the application of Mercer’s condition, which means that

\[
K(x_i, x_j) = \varphi(x_i)^T \varphi(x_j)
\]

3. Image denoising based on image approximation via SVR

Gray level image can be regarded as a two-dimensional continuous function,

\[
y = f_{image}(x)
\]

Where input \( x \in R^2 \) is a two-dimensional vector that indicates the position of a pixel, where output \( y \in R \) is a scalar value denoting the gray level of that pixel. Each pixel could be a training data. If the width of the image is \( M \) and height is \( N \), then the number
of training examples is $M \times N$. According to equation (1) and (6), the image could be represent as

$$f_{\text{image}}(\mathbf{x}) = \sum_{i=1}^{M} \sum_{j=1}^{N} (\alpha^*_ij - \alpha_{ij}) K(x_{ij}, \mathbf{x}) + b \tag{7}$$

Image approximation via SVR could reduce two kinds of noise, Gaussian random noise and salt-and-pepper noise. Gaussian random noise can regard as the little distortions below or above the image gray level, and salt-and-pepper noise means the pixels whose gray level have been totally destroyed. According to equation (3), we can find that the insensitivity region $\varepsilon$ and the positive real constant $c$ are useful for us to remove the noise. $\varepsilon$ allows training error to be within the range $\pm \varepsilon$. Therefore, Gaussian random noise within this range can be smoothed by adjusting the value of $\varepsilon$. The value of $c$ is used to adjust the amount of outliers. We can select $c$ to a small value so that the salt-and-pepper noise is regarded as outliers, the image function will not approximate their value accurately and the salt-and-pepper noise could be removed. Figure 1 explains how to remove the two kinds of noises by image approximation via SVR. The example is 1-dimension signal $y = f(\mathbf{x})$, where $\mathbf{x}$ denotes scalar input. In this case, we make $\varepsilon = 0.3$, $c = 5$ and use Gaussian RBF [3] with $\sigma^2 = 0.8$ as the SVR kernel. Gaussian RBF could be written as follow:

$$K_{\text{RBF}}(\mathbf{x}, \mathbf{x}') = \exp\left\{-\frac{\|\mathbf{x} - \mathbf{x}'\|^2}{\sigma^2}\right\} \tag{8}$$

The black asterisks are sample noisy data. The noises are Salt-and-pepper noise and Gaussian noise with mean 0 and standard deviation 0.3. The blue curve is the result of approximation via SVR. The two red dotted curves lying above and below the middle blue curve are the walls of the insensitivity tube. Most samples are lying within the insensitivity tube, which removes the random noise. On the other hand, the value of $c$ is small enough to make the result approximate the signal accurately and remove the salt-and-pepper noise.

4. Wavelet kernel and Wavelet support vector machine

Function $\psi(\mathbf{x}) \in L^2(R)$ could be a mother wavelet if it satisfies the condition below [4],

$$c_\psi = \int_{-\infty}^{+\infty} \frac{\hat{\psi}(\omega)}{|\omega|} \; d\omega < +\infty \tag{9}$$

Where $\hat{\psi}(\omega)$ is Fourier transform of $\psi(\omega)$. The wavelet transform of a function $f(\mathbf{x}) \in L^2(R)$ can be written as

$$W_f(a,b) = \left\langle \psi_{(a,b)} \cdot f(\mathbf{x}) \right\rangle \tag{10}$$
Image Denoising Based on Wavelet Support Vector Regression

Fig. 1. The example for 1-dimension signal SVR

Where \( a \neq 0 \) is a dilation factor, \( b \in R \) is a translation factor and \( \psi_{(a,b)}(x) \) is

\[
\psi_{(a,b)}(x) = \frac{1}{\sqrt{|a|}} \psi \left( \frac{x - b}{a} \right)
\]

(11)

The decomposition of \( f(x) \) on a wavelet basis \( \psi_{(a,b)}(x) \), \( f(x) \) can be reconstructed as follow:

\[
f(x) = \frac{1}{c_\psi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_f(a,b) \psi_{(a,b)}(x) da/a^2 db
\]

(12)

A function could be express by a family of functions generated by dilation and translation of mother wavelet. If we take finite terms to approximate \( f(x) \), then

\[
f^\hat{}(x) = \sum_{i=1}^{N} W_f(a_i,b_i) \psi_{(a_i,b_i)}(x)
\]

(13)

where \( f(x) \) is approximated by \( f^\hat{}(x) \). The multidimensional wavelet function could be written as [5]

\[
\psi_d(x) = \prod_{i=1}^{d} \psi(x_i)
\]

(14)

Fig. 2. Original Image

Where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \). Let’s consider the Morlet mother wavelet as below

\[
\psi(x) = \cos(1.75x) \exp(-\frac{x^2}{2})
\]  

(15)

the multidimensional wavelet function is

\[
\psi(x) = \prod_{i=1}^{d} \exp \left[\frac{- (x_i - b_i)}{2a_i^2}\right] \cdot \left[ \frac{1.75 (x_i - b_i)}{a_i} \right]
\]  

(16)

We can construct translation invariant wavelet kernel

\[
K_w(x - x') = \prod_{i=1}^{d} \left[ \cos 1.75 \frac{(x_i - x'_i)}{a_i} \right] \cdot \exp \left[\frac{- (x_i - x'_i)/2a_i^2}{2}\right]
\]  

(17)

\( K_w(x - x') \) is an admissible support kernel [6]. And we can construct the wavelet support vector machine (WSVM) via \( K_w(x - x') \). With the wavelet kernel, the result of approximation of function \( f(x) \) using WSVM could be written as

\[
f(x) = \sum_{i=1}^{N} (\alpha_i^* - \alpha_i) K_w(x_i, x) + b
\]  

(18)

where \( N \) denote the number of training data.

5. Results of experiment

In this section, we add salt-and-pepper noise and Gaussian random noise with mean 0 and standard deviation 1 to the image with size 100 × 100.

We process the noisy image with the Gaussian RBF SVM, WSVM and other traditional method. Figure 2 and figure 3 shows the original image and the noisy image,
Image Denoising Based on Wavelet Support Vector Regression

Fig. 3. Noisy Image

Fig. 4. Result of WSVM

Fig. 5. Result of Gaussian RBF SVM

Fig. 6. Result of Average Filtering
Figure 4 and figure 5 show the result of denoising via WSVM and SVM, Figure 6, Figure 7 and figure 8 shows the result of gaussian filtering, average filtering and medium filtering. We define the image signal noise ratio (SNR) as follow:

$$SNR_{image} = \sqrt{\frac{\sum_{i=1}^{M} \sum_{j=1}^{N} [f(i,j)]^2}{\sum_{i=1}^{M} \sum_{j=1}^{N} [f(i,j) - \hat{f}(i,j)]^2}}$$  \hspace{1cm} (19)

Where $f(i,j)$ is the original image and $\hat{f}(i,j)$ is the result of image denoising, the resolution of image is $M \times N$. Table 1 lists the parameters and SNR for each method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>SNR</th>
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</thead>
<tbody>
<tr>
<td>WSVM</td>
<td>$a = 1$</td>
<td>6.2072</td>
</tr>
<tr>
<td>Gaussian RBF SVM</td>
<td>$c = 10, \sigma^2 = 0.05$</td>
<td>5.3094</td>
</tr>
<tr>
<td>Average filtering</td>
<td>Size of filter:33</td>
<td>5.0564</td>
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<tr>
<td>Medium filtering</td>
<td>Size of filter:33</td>
<td>4.9103</td>
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<tr>
<td>Gaussian filtering</td>
<td>Size of filter:33</td>
<td>5.1899</td>
</tr>
</tbody>
</table>

6. Conclusion and discussion

In this paper, the function approximation via SVR is reviewed based on which we analyze the model of noise and discuss the feasibility of denoising via SVR. Wavelet theory is briefly discussed and the wavelet support vector machine (WSVM) is constructed based on wavelet kernels. At last, we process the noisy image with WSVM and other image denoising method, which indicate that the WSVM could remove the random noise and salt-and-pepper noise better than Gaussian RBF SVM and other traditional method.

7. Reference

References

1998

1998

2000

1990

1992

2004